II

ANALYSIS SITUS

ANALYSIS Situs or Topology—the two words are used interchangeably at present—has its roots in a series of physical questions. To mention only one such question, how do you distinguish the following three knots?

Suppose they are made of rubber tubing and can be deformed without tearing or otherwise doing anything discontinuous. The problem is to learn how to state properties which are not altered by suitably defined continuous transformations and which will distinguish non-equivalent figures from each other.

As an independent science Analysis Situs is a product of the present century. Individual results had been obtained previously by various mathematicians, as for example Euler, Riemann, Möbius, Betti, and Dyck. Physicists had also used elementary topological considerations. For example, Kirckhoff used them in his theory of the flow of current through a network of wire, Maxwell used them in the induction of linking circuits, and Thompson and Tait had imagined that the different atoms could be treated as vortices which are knotted in various ways. But it was the work of Poincaré at the opening of the present century that gathered together the scattered beginnings and brought to light a large body of new theorems and problems which evidently constitute an important mathematical discipline. The subject has developed with extraordinary rapidity—comparable only with one or two branches of mathematical physics in this respect—and the mathematicians of the United States have played a leading part in this development. The account of the subject which one would get by describing the work of three or four of our colleagues would be quite an adequate one. But, as I said at the beginning of these lectures, any localized "school" of mathematics at the present time is bound to have a very transitory existence, and any attempt to treat such a grouping of men as a distinct entity seems to me to be rather futile. Therefore I shall try to arrange what I have to say in terms of the natural subdivisions of the science itself.

The central trunk of the tree of Analysis Situs, from which all the other branches grow out in various directions, is the topology of an arithmetic space of n dimensions. This space is simply the set of all ordered sets of n numbers $(x, y, z, \ldots, t)$ and its topology is the set of properties which are unaltered by the group of all single valued continuous transformations with single-valued inverses. Such transformations are called *homeomorphisms*.

The topology of this space is a branch of analysis, for its theorems are theorems about the real number system. But they can also be regarded as theorems about any space in one-to-one reciprocal continuous correspondence with ("homeomorphic" with) the arithmetic space. Thus they are theorems about a Euclidean space of n dimensions or about the interior of a sphere in a Euclidean space of n dimensions, for both of these figures are spaces homeomorphic with the arithmetic space of n dimensions. The general term for such a space is an "n-cell" and an

n-cell may be defined by means of the following set of axioms.[1]

The undefined terms are "points" and "allowable coordinate systems." The axioms are:

1. An allowable coordinate system is a one-to-one reciprocal correspondence $P \rightarrow x$ between the totality of points and the totality of ordered sets of n real numbers $(x_1, \ldots, x_n)$.
2. If $P \rightarrow x$ and $P \rightarrow y$ are two allowable coordinate systems, the "transformation of coordinates" $x \rightarrow y$ determined by them is one-to-one, reciprocal, and continuous.
3. If $P \rightarrow x$ is an allowable coordinate system and $x \rightarrow y$ a one-to-one reciprocal continuous transformation of all sets of real numbers $x_1, \ldots, x_n$, then the resultant of $P \rightarrow x$ and $x \rightarrow y$ is an allowable coordinate system $P \rightarrow y$.
4. There is at least one allowable coordinate system.

These axioms determine that the group of transformations of allowable coordinate systems into allowable coordinate systems is of the form

$$y_i = f_i(x_1, \ldots, x_n)$$

where the functions are continuous for all values of x and such that each transformation has a continuous inverse. The group of the n-cell is isomorphic with this group. The analysis situs of the n-cell is the theory of those of its properties which are invariant under this group.

This group obviously has the Euclidean group as a subgroup. Hence all the theorems of the analysis situs of an n-cell are theorems of Euclidean geometry, but, of course, not conversely. For example, if $n=2$, there is the theorem of Jordan that a simple closed curve separates the plane into two regions. This is a theorem of Euclidean geometry,

[1] *Cf.* Chap. III, §7 of the forthcoming Cambridge Tract by Whitehead and myself to which I referred in the first lecture.

but everything about it is invariant under the topological group and so it is also a theorem of analysis situs.

As another example, let $n=1$, the n-cell now being a linear continuum, and consider the theorem that the complementary set of any closed set is a set of open segments. This is a theorem about the Euclidean geometry of one dimension and also about the topology of a 1-cell. On the other hand, the theorem that any closed set has a measure is not a theorem of analysis situs because measure is not left invariant by the transformations of the topological group.

Any theorem of the analysis situs of a 1-cell can be regarded as a theorem of analysis. For the set of all real numbers is a 1-cell. The identity is a sub-group of the analysis situs group. Analysis is the geometry of this 1-cell under this group. For analysis cherishes the distinction between each number and each other number, whereas most of the geometries that we recognize as such treat all points alike.

The two examples I have mentioned are perhaps enough to illustrate the fact that a large portion of the point-set theory which has been developed in the last fifty or sixty years is analysis situs. The theorems of this part of analysis situs are stated to a large extent in terms which have a meaning as well when applied to a 1-cell as when applied to an n-cell. I am referring to such concepts as limit point, cluster point, neighborhood, closed set, open set, boundary, perfect, compact, complete, separable, connected, locally connected, prime part, arcwise connected, and so on—all ideas which arise in linear or planar point-set theory. This part of analysis situs has been cultivated with signal success during the last two decades by your near neighbor, Professor R. L. Moore, and his students, as well as by many other mathematicians in other parts of the world. The work of this group may be characterized as taking these concepts of point-set theory, sharpening all the distinctions as much as possible, and recombining them in all possible ways to characterize point sets of very general types. A typical result is the theorem of Hans Hahn that a necessary and sufficient condition that a continuous curve can be drawn through a closed, compact, perfectly separable, connected point set is that the set shall be locally connected. (Locally connected means that for every point P and positive number ϵ there exists a positive δ such that if x and y are two points at a distance from P less than δ, they lie in a closed connected subset which is within ϵ of P.) Other typical results are the theorems of Hahn, Moore and others about the sets of points which can be regarded as curves in terms of their "prime parts."

It is a very short step from the theorems of this class to the theory of abstract spaces such as were brought to the attention of the mathematical public by the Paris thesis of Fréchet in 1906, and which were actively studied by E. H. Moore and his students at the same epoch. Among these spaces are what are called topological spaces. Such a space may be defined by the set of four axioms in terms of the concepts, point and neighborhood, given by Hausdorff in his book on *Mengenlehre* in 1913. (A similar but less elegant set of axioms was given by R. E. Root in 1910.)

These axioms state properties of points and neighborhoods that are obviously true of points and neighborhoods in ordinary space, and from which it is possible to deduce a large class of theorems about limit points, contact, continuous functions, continuous transformations, and so on. It is also true that the class of spaces which satisfy these axioms is very broad, so that it includes a very large proportion of the generalized spaces which are used in present-day mathematics.

There is a definition of what is meant by a homeomorphism between any two topological spaces. Hence there is a theory of the conditions under which such homeomorphisms can take place. This theory is topology or analysis situs in the broadest sense in which the words are used today.

There is also the theory of those properties of a particular topological space which are left invariant by the group of all homeomorphisms of this space with itself. This is the analysis situs of a particular topological space. A special instance is the topology of an n-cell.

At first sight one would think that the topological spaces are so general that very little can be said about them. But if you will think over the list which I mentioned a few minutes ago of concepts which arise by generalization from the theory of linear point-sets, you will see that there must be a very large and interesting set of theorems about the interrelations of these concepts. For example, there is the theorem that every connected, locally connected, complete metric space is arcwise and locally arcwise connected. ("Complete" means for a metric space that each Cauchy sequence has a limit point.) This theorem, stated by Menger (1929) in this form, is derivable from a somewhat more general theorem of R. L. Moore (1927). Also there are the theorems such as those of Chittenden and Urysohn about the conditions under which a topological space is metrisable (under which a definition of distance satisfying certain standard conditions can be introduced).

Perhaps the most interesting chapter in this generalized analysis situs is the theory of dimensionality of Urysohn and Menger. This theory succeeds in giving a definition, in terms of the concepts of neighborhood and boundary, which attaches a definite integer (which may be zero) to each point of a topological space. Moreover this integer is n for each point of an n-cell and it satisfies a considerable number of theorems which a dimensionality number should satisfy. Obviously there is no room in this sort of a rapid summary for a résumé of these theorems. Let us be content with the paraphrase of the definition due, I think, to Brouwer (who himself profoundly influenced the development of the theory): A space is n-dimensional at a point P if the walls of any prison to confine P are $(n-1)$-dimensional. The chapter on dimensionality seems to have endowed the theory of generalized spaces with a substantial quality which places it among the classical branches of mathematics.

From these very general considerations we can specialize down to what we may call the theory of "regular n-dimensional manifolds." Every point of such a manifold has a neighborhood homeomorphic with an n-cell. This class of manifolds includes all curves, surfaces, and k-dimensional varieties, which occur in the theory of an arithmetic n-space, provided we exclude boundary and singular points.

As these manifolds come to us in ordinary geometry or analysis they have certain additional attributes of smoothness which are not, strictly speaking, topological properties. What is relevant topologically is that each point is surrounded by an "n-cell" which is in (1-1) correspondence with an n-cell in the arithmetic space of n dimensions, and that these "n-cells" overlap each other in a manner which is not too bizarre. The (1-1) correspondence $P \rightarrow x$ between the points P, of an "n-cell" of our manifold and the points $x=(x^1, \ldots, x^n)$ of an n-cell of the arithmetic space is essentially a coordinate system in the sense of analytic geometry. For a coordinate system is merely an association of each point P with a set of n numbers $(x^1, \ldots, x^n)$.

Since the manifolds which we are here calling "regular" are all topological spaces, they may be characterized by adding certain other axioms to the Hausdorff set. They have also been described, without presupposing the theory of topological spaces, in a set of axioms which makes the notion of "allowable coordinate systems" fundamental, in Chapter VI of the Cambridge Tract to which I have already referred more than once. The axioms are referred to in this book as the axioms of differential geometry, and they provide for a degree of "smoothness" corresponding to the number of derivatives possessed by the most general functions allowed in transformations of coordinates. When these transformations are of "class zero," i.e., continuous but not necessarily endowed with derivatives, we have the "regular n-dimensional manifolds" of topology in the sense in which I am using the words.

I refer to this formulation of the theory of regular manifolds with so much emphasis because it seems to bring to light what I think is one of the most important unsolved mathematical problems of our epoch—the relation between differential geometry and topology. Differential geometry, in a sense which I hope to make a little more precise in my lecture tomorrow, describes the "local structure" of a regular n-dimensional manifold. The general problem to which I refer is: How are the topological properties of the manifold restricted by specifying the local structure? Progress on this problem has been recently made by Hopf in the theory of locally flat spaces and by Morse in his calculus of variations in the large.

A fundamental question in the theory of regular manifolds is the triangulation problem. Any two-dimensional regular manifold can be decomposed into a system of cells which are images of triangles and which meet each other only in edges and vertices. The problem is, can an analogous subdivision be made of a three-dimensional regular manifold into tetrahedra, and of an n-dimensional one into simplexes (generalized triangles). This question has been answered in the affirmative by van der Waerden for manifolds given by algebraic relations in a Euclidean space, and a similar result where the relations are simply differentiable has been announced by S. S. Cairns. The problem for regular manifolds of class zero (where differentiability of the functions which enter in the coordinate transformations is not assumed) is much more difficult.

The question of triangulation is important because its solution would show the exact relationship between the theory of regular manifolds and what I think may best be called "classical analysis situs." This is the part of the subject which is often referred to as combinatorial analysis situs, but the adjective "combinatorial" would, I think, better be reserved for the theory which has been obtained by abstracting the genuinely combinatorial elements from the classical analysis situs and which has grown into a distinct mathematical discipline in the hands of Alexander, Newman, and van Kampen.

By a *complex* let us mean any topological space which is in (1-1) reciprocal and continuous correspondence with an n-dimensional polyhedron in a Euclidean space of any number of dimensions. A polyhedron is to be understood in the broadest sense as a figure made up of a finite number of finite flat n-cells with flat boundaries, no two of the n-cells having any interior point in common. The theory of such complexes is what I mean by classical analysis situs.

For a carefully formulated definition of a complex I refer you to the volume on *Topology* by Lefschetz in the Colloquium series of the American Mathematical Society. The definition is stated somewhat differently in my volume on *Analysis Situs* in the same series, since I did not presuppose the idea of a space in which limit points are defined, but used something very like the "allowable coordinate system" scheme which appears in Whitehead's and my axioms referred to earlier today. The complex which is obtained as the field of operation for classical analysis situs is the same both in Lefschetz's formulation and in mine. This analysis situs is combinatorial in the sense that the topological properties of the complex are all determinate as soon as a finite number of conditions are given which specify how the n-cells are joined together. These conditions can be embodied in a finite number of finite matrices.

By operations with these matrices one may arrive at various constants or invariants of the complex and at various identities which relate these invariants. But it is obvious that the same complex, as a set of points, can be decomposed in infinitely many ways into cells. Hence it is necessary, in order to establish any invariant as such, to show that it is independent of the choice of the particular cellular structure by which it is defined. This "invariance proof" is apt to make use of continuity arguments quite as deep as any used in point-set theory. Thus the classical analysis situs is by no means merely a branch of combinatorial analysis.

The chief instrument of classical topology is the p-chain, which is a combination of p-cells each associated with a number. If the p-cells are denoted by $\overset{i}{E}_p$ and the associated numbers by k_i, then the p-chain is denoted by

$$C_p = \sum_{i=1}^{\alpha_p} k_i \overset{i}{E}_p$$

where α_p is the total number of p-cells in a given cellular subdivision of our n-dimensional complex. The chains can obviously be added after the fashion of the elements of a linear associative algebra and, indeed, form a commutative group with respect to addition.

The boundary of a p-cell is a $(p-1)$-chain in which the coefficients are $+1$, -1, or 0. This permits one to define what is meant by the boundary of any p-chain, namely, as the linear combination, where B_p^i is the boundary of E_p^i. In the particular case in which this boundary vanishes, the p-chain is what is called a p-cycle. A p-cycle is essentially a closed p-dimensional variety which may have singularities. In particular, the boundaries of $(p+1)$-chains are all p-cycles—they are called bounding p-cycles and also are said to be homologous to zero.

With respect to the operation of addition, the set of all p-cycles of a given cellular structure constitutes a commutative group. If the coefficients k_i which are admitted are integers (they might be marks of any field whatever) this is an infinite group with a finite basis. The bounding p-cycles form a subgroup of this group. The quotient group of the whole group of p-cycles by this subgroup is an invariant of the original k-dimensional complex, i.e., the same abstract group is obtained by this process no matter what subdivision of the complex into cells is employed.

This quotient group is called the p-dimensional homology group or Betti group of the n-dimensional complex. The well-known arithmetic invariants of this group are the p-dimensional Betti numbers and coefficients of torsion of the complex. Any p-chain of the bounding subgroup is said to be homologous to zero. The p-chains in any co-set (in the group-theoretic sense) of this subgroup are said to be homologous to each other. Two p-chains in different co-sets are not homologous.

What I have just been stating is the modern version[1] of

[1] *Cf.* Pontrjagin, *Math. Ann.*, Vol. 105.

Poincaré's theory of homology. If one is allowed to presuppose the theory of Abelian groups (as I have been doing for the last few minutes) you see that one may penetrate very rapidly into the very heart of the classical analysis situs. Poincaré's contribution to the subject was the general formulation of the homology theory, the discovery of the numerical invariants referred to above, and two identities which they satisfy, now known as the Euler-Poincaré relation and the Poincaré duality relation. The first of these relates the alternating sum of the numbers of cells of all dimensionalities in a given cellular subdivision with the alternating sum of the Betti numbers; the second is an equality between the p-dimensional and $(n-p)$-dimensional invariants of a manifold.

Of the further development of the homology theory I think it is fair to say that the two major achievements are Alexander's duality theorem and Lefschetz's theory of intersections and fixed points.

Alexander's theorem is a broad extension of the Jordan theorem that a simple closed curve separates a Euclidean plane into two simple regions. If a complex L of p dimensions is contained in a space S of n dimensions, the theorem states an equality between the invariants of L and those of the complementary space $S-L$. This theorem has been extended by Alexandroff, Frankl, Lefschetz, and Pontrjagin to cover the cases in which L is an arbitrary closed set.

The Lefschetz intersection theorems describe the nature of the intersection of a p-dimensional with a k-dimensional chain in an n-dimensional manifold. These results have been applied by Lefschetz himself to determine important formulas for the number of fixed points of continuous transformations of a complex into itself and also by Lefschetz and van der Waerden to the solution of one of the riddles of algebraic geometry, a rigorous foundation and formulation of the enumerative geometry.

These and other successes of the classical analysis situs have been extended by various ingenious processes of approximation to topological spaces of an extremely general type. I am referring to the work of Alexandroff, Vietoris, and others, as a result of which something like a combinatorial "skeleton" can be seen giving form to these spaces which at first seemed so hopelessly amorphous. Indeed, Alexandroff has succeeded in obtaining a theory of dimensionality by a limiting process from the combinatory homology theory.

On the other hand, the classical analysis situs itself is very far from having solved its own fundamental problems. For $n > 2$ there is nothing in existence which resembles a complete set of invariants. Even for $n = 3$ there is no set of invariants known whose vanishing will require a three dimensional manifold to be homeomorphic with a 3-sphere. As a consequence of this, it is not known how to give for $n = 4$ the combinatorial criteria which state that a four-dimensional complex is a manifold (e.g., that it be a regular manifold as the latter is defined above).

Roughly speaking, the part of classical analysis situs which depends on elementary properties of Abelian groups, seems to be in good order and the direction of attack on its fundamental unsolved problems seems to lead into the deeper problems of discrete group theory. Here one may think of the work of Alexander and Reidemeister on the theory of knots and of Nielson and others on the enumeration of fixed points of transformations beyond the algebraic count given by Lefschetz.

The theory of fixed points of transformations of topological spaces is important in the work of Birkhoff and Morse and others on problems like those of the singularities of vector fields. For making no attempt to fit these researches, still in progress, into an orderly classification of the science of topology there are two good reasons—the certainty that I am not competent, and the probability that the science is not yet far enough developed.